J. DIFFERENTIAL GEOMETRY
31 (1990) 847-857

ESTIMATING $\|d\varphi^t\|$ FOR UNIT VECTOR FIELDS WHOSE ORBITS ARE GEODESICS

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Introduction

In the following, all manifolds, vector fields, etc., will be assumed to be real analytic. Let M be a connected, n-dimensional, complete riemannian manifold, and v a unit vector field (i.e., $|v| \equiv 1$) all of whose orbits are geodesics of M (i.e., $\nabla_v v \equiv 0$).

Although it is perhaps not really necessary, we also assume that *not all orbits of* v *are closed*, otherwise, by Wadsley's Theorem [11], there exists an S^1-action on M with the same orbits as v, and our problems should probably be studied in that context.

At each point $x \in M$ define $e_x = \max\{|\nabla_z v|^2 - K_{zv}\}$, where z ranges over all unit vectors $z \in T_xM$ perpendicular to v; here K_{zv} denotes the sectional curvature of M at x with respect to the 2-plane spanned by z and v.

Let φ^t be the flow generated by v and, for each $x \in M$ and any $t \geq 0$, define $E_{xt} = \max e_y$, where y ranges over the orbit interval $[\varphi^{-\sqrt{2}t}(x), \varphi^{\sqrt{2}t}(x)]$.

Theorem II. *Assume* $x \in M$ *is such that* $e_x \geq 0$ (*for example, at* x *suppose* $K_{zv} \leq 0$ *for some* z *as above*). *Then for any unit vector* $u \in T_xM$ *and all* $t \geq 0$ *we have*

$$|d\varphi^t(u)|^2 + |d\varphi^{-t}(u)|^2 \leq 2\cosh^2 t\sqrt{E_{xt}}.$$

Examples. (i) In the (trivial) case when v is also a Killing vector field, it is easy to see that $e_x \equiv 0$ and our inequality is sharp in this case (see §5).

(ii) If v is the geodesic flow on the unit sphere bundle SM^2 of a surface M^2, and we consider the curvature as a function $K: M \to R$, then at the point $x = (y, \xi_y)$ of SM^2

$$2e_x = (K-1)^2 + [(K^2-1)^2 + (dK(\xi))^2]^{1/2} \geq 0$$

Received December 5, 1988.

and the above inequality holds with the corresponding E_{xt} (see §5).

Theorem II is an immediate corollary of a sharper inequality (see Theorem I, below) which however involves terms depending on $d\varphi^t$ on the right side, i.e., 'dynamic' terms, which do not seem to have an immediate 'static' expression like the quantity e_x above.

We also obtain a lower bound (see Theorem III of §4) from which as an immediate corollary we obtain

Theorem IV. *Let* $e'_x = \min\{-K_{zv}\}$ *where* z *ranges over all unit vectors perpendicular to* v, *and let* $E'_{xt} = \min e'_y$, *where* y *ranges over the orbit interval* $[\varphi^{-\sqrt{2}t}(x), \varphi^{\sqrt{2}t}(x)]$. *If* $E'_{xt} \geq 0$, *for any unit vector* $u \in T_xM$ *and all* $t \geq 0$, *then we have*

$$|d\varphi^t(u)|^2 + |d\varphi^{-t}(u)|^2 \geq 2\cosh^2 t\sqrt{E'_{xt}}.$$

Notice as a corollary, via the Liapunov exponents, one obtains bounds on the entropy of φ^t.

We thank K. Grove and the referee for helpful comments.

1. Statement of Theorem I

Denote by D the diagonal of $M \times M$ (with the product riemannian metric). On $M \times M$ define a unit vector field $V_{(x,y)} = 2^{-1/2}(v_x, -v_y)$, and let φ^t and Φ^t be the flows generated by v and V on M and $M \times M$, respectively.

Given $x \in M$ and t we define a real number as follows:

$$\bar{e}_{xt} = \max_{\overline{Z}}\{|\nabla_{\overline{Z}}V|^2 - K_{\overline{Z}V} - |\beta(\overline{Z}, V)|^2\},$$

where $\overline{Z} = d\Phi^t(Z)/|d\Phi^t(Z)|$ with $Z \in T_{(x,x)}D$, $\beta(\overline{Z}, V)$ is the orthogonal projection of $\nabla_{\overline{Z}}V$ into the n-plane $d\Phi^t(T_{(x,x)}D)$, and $K_{\overline{Z}V}$ is the sectional curvature of $M \times M$ at $\Phi^t(x, x)$ with respect to the 2-plane generated by $\overline{Z}$ and V.

Define $\overline{E}_{xt} = \max \bar{e}_{x\tau}$ for $\tau \in [0, t]$ and assume $\overline{E}_{xt} \geq 0$.

Theorem I. *For any* $x \in M$ *and any unit vector* $Z \in T_{(x,x)}D$ *we have* $|d\varphi^t(Z)| \leq \cosh t\sqrt{\overline{E}_{xt}}$ *for all* t; *i.e., since* $\Phi^t = (\varphi^{t/\sqrt{2}}, \varphi^{-t/\sqrt{2}})$, *for every unit vector* $u \in T_xM$ *we have*

$$|d\varphi^t(u)|^2 + |d\varphi^{-t}(u)|^2 \leq 2\cosh^2 t\sqrt{2\overline{E}_{x\sqrt{2}t}} \quad \textit{for all } t.$$

Theorem II is obtained by simply dropping the β term above.

The number $\bar{e}_{xt}$ can also be obtained as follows: Given $x \in M$ let $\{b_i\}$ be any (not necessarily orthonormal) basis of T_xM. Then for each real t define an $n \times n$ matrix B_t by

$$B_t = \{\langle d\varphi^{t/\sqrt{2}}(b_i)\, d\varphi^{t/\sqrt{2}}(b_j)\rangle\},$$

and set $\mathscr{B}_t = B_t + B_{-t}$. Thus $2\bar{e}_{xt}$ is equal to the maximum eigenvalue of $\ddot{\mathscr{B}}_t\mathscr{B}_t^{-1} - \frac{1}{2}(\dot{\mathscr{B}}_t\mathscr{B}_t^{-1})^2$.

Our proof of Theorem I simply consists of applying a version of the Rauch Comparison Theorem (as stated on p. 188 in [3]) to orbits of the vector field V suitably "lifted' to the graph, $\mathfrak{G}(v)$, of the 1-foliation defined by v on M (see §2).

The number $\bar{e}_{xt}$ is the maximum of the negative of the sectional curvature at $\Phi^t(x, x)$ of $\mathfrak{G}$ with respect to all 2-planes containing V.

2. The associated vector field $\overline{V}$

In this section we substitute the study of v on M by the study of an intimately related vector field $\overline{V}$ defined on the so-called graph, $\mathfrak{G} = \mathfrak{G}(v)$, of the 1-foliation defined by the vector field v. $\overline{V}$ will also be a unit vector field whose orbits are geodesics of $\mathfrak{G}$, but in addition it will be the gradient of a certain riemannian submersion $\delta\colon \mathfrak{G} \to R$. Although perhaps this change of scenario is not so interesting in a purely topological way, it is nontrivial in the *differential geometric* sense.

Recall (see [12], [9]) that $\mathfrak{G} = \mathfrak{G}(v)$ consists of all triples $(x, y, [\alpha])$, where x and y lie on the same orbit, γ, of v, and $[\alpha]$ is an equivalence class of arcs α contained in γ; two arcs α, β are equivalent if the (foliation theoretic) holonomy along $\alpha\beta^{-1}$ is the identity. Hence, if v has no closed orbits as a set, $\mathfrak{G}$ simply consists of the subset $\{(x, y) \in M \times M | x$ and y lie on the same orbit of $v\}$, and it is convenient at first to go through our arguments by assuming this is the case.

If the holonomy of v is real analytic, then $\mathfrak{G}$ is an $(n+1)$-manifold in a natural way [12, p. 62], and one has a canonical immersion $I\colon \mathfrak{G} \to M \times M$ defined by $I((x, y, [\alpha])) = (x, y)$ and two submersions $p_1, p_2\colon \mathfrak{G} \to R$ defined by $p_1((x, y, [\alpha])) = x$ and $p_2((x, y, [\alpha])) = y$.

We make $\mathfrak{G}$ into a riemannian manifold by requiring that I be an isometric immersion, i.e., we pull back the product metric of $M \times M$ via I.

Notice that the diagonal D (of $M \times M$) is a totally geodesic submanifold of $\mathfrak{G}$.

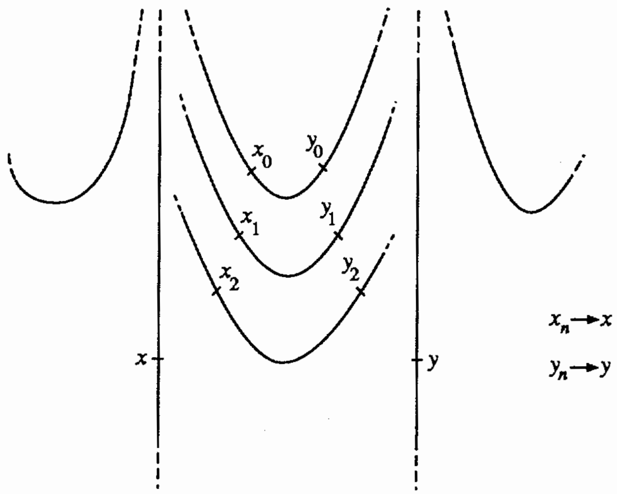

FIGURE 2.1

Proposition 2.1. *If the orbits of v are geodesics, then this riemannian metric of $\mathfrak{G}(v)$ is complete for complete M.*

Remark. It is necessary that the orbits be geodesics.

Consider the one-foliation of R^2 shown in Figure 2.1. The sequence $z_n = (x_n, y_n) \in \mathfrak{G}$ shown is a divergent Cauchy sequence in $\mathfrak{G}$, since the limits x and y lie in different orbits.

Proof of Proposition 2.1. Let $(x_n, y_n, [\alpha_n])$ be a Cauchy sequence in $\mathfrak{G}$. Then it is easy to see that x_n and y_n are Cauchy sequences in M, which converge to x and y respectively, since M is complete.

It is enough to show that x and y lie on the same orbit, and for this to happen it is enough to show that the lengths of the orbit segments s_n from x_n to y_n remain bounded for all n.

Let $a_n(t)$ and $b_n(t)$, $t \in [0, 1]$, be two smooth arcs in M from x_0 to x_n and y_0 to y_n, respectively, such that for all $t \in [0, 1]$, $a(t)$ and $b(t)$ lie on the same orbit, and let Σ denote the two-dimensional surface of M consisting of the union of the orbit arcs s_t from $a(t)$ to $b(t)$ for all $t \in [0, 1]$ (see Figure 2.2). Let w be the dual 1-form of v. Then (see Wadsley [11, p. 542]) $dw(\ , v) = 0$, and so $dw = 0$ on the 2-plane fields tangent to the surface Σ.

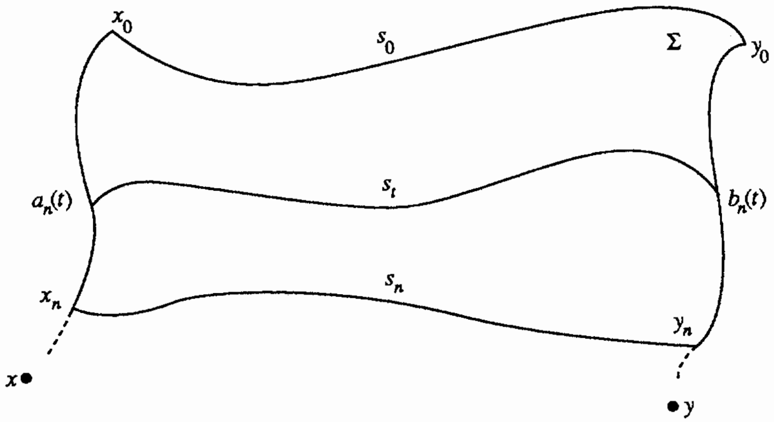

FIGURE 2.2

Applying Stokes' Theorem we obtain

$$\int_a w + \int_{s_0} w + \int_{s_n} w + \int_b w = \int_\Sigma dw = 0,$$

which shows the lengths of the orbit segments s_n are bounded for all n and Proposition 2.1 is proved.

Proposition 2.2. *If not all orbits of v are closed, then no closed orbit γ can have (foliation theoretic) holonomy of finite order.*

Proof.[1] Let $\varphi^t(x)$ denote the flow of v. If γ_x, of length c, were such an orbit, it would follow from Wadsley's proof [8, Corollary 4.4 and the fact $B_1 = \varnothing$] that $\varphi^c(x)$ is the identity in a neighborhood of γ_x and hence, since v is real analytic, it is the identity on all of M, i.e., all orbits would be closed and of length c.

Let $\overline{V}$ denote the unique C^ω vector field of $\mathfrak{G}$ defined at $(x, y, [\alpha])$ by $dp_1(\overline{V}) = v_x/\sqrt{2}$ and $dp_2(\overline{V}) = -v_y/\sqrt{2}$, where p_1 and p_2 are the natural projections of $\mathfrak{G}$ into M, i.e., $dI(\overline{V}) = V$.

Proposition 2.3. *There exists a riemannian submersion $\delta\colon \mathfrak{G} \to R$ such that $\overline{V} = \operatorname{grad}\delta$.*

Proof. Let U be a neighborhood of the diagonal D in $M \times M$ which is so small that the function $\delta_0\colon U \to R$ defined by $\delta_0(x, y) =$ oriented

[1] We thank Y. Carrière for this short proof.

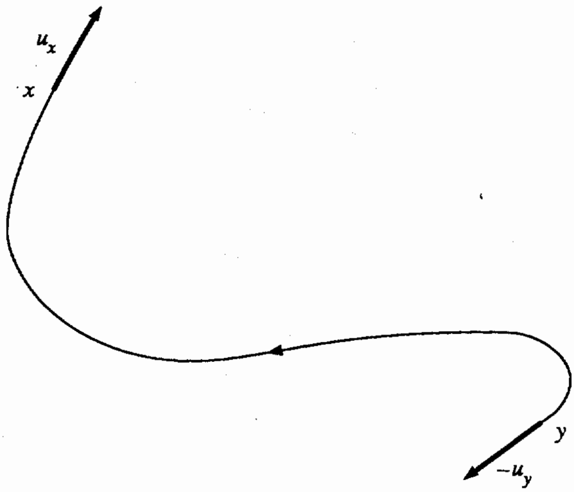

FIGURE 2.3

distance in M from x to y (with respect to the orientation of v) is well defined. By the Gauss Lemma applied to D, the gradient of δ_0 coincides (up to sign) with the vector $(u_x, -u_y)/\sqrt{2}$ of $M \times M$ at (x, y), where u_x and u_y are unit vectors of M tangent at x and y to the unique minimizing geodesic segment of M from x to y (see Figure 2.3).

Let U_0 denote a small enough neighborhood of D in $\mathfrak{G}$. Since the orbits of v are geodesics of M, the vector field $\overline{V}$ of $\mathfrak{G}$ restricted to U_0 coincides (up to sign) with the gradient of the function $\delta = \delta_0 \cdot I \colon U_0 \to R$, where $I \colon \mathfrak{G} \to M \times M$ is the natural isometric immersion. Since the orbits of V are geodesics in $M \times M$, the orbits of $\overline{V}$ are geodesics in $\mathfrak{G}$ and so the function δ is a *riemannian* submersion on U_0 (see [9, p. 155]).

Since U_0 is open in $\mathfrak{G}$ and $\overline{V}$ is real analytic, the same holds in all of $\mathfrak{G}$, where now the function δ is defined at any $(x, y, [\alpha]) \in \mathfrak{G}$ as the oriented distance in M between x and y along the orbit γ with α indicating (in the case γ is closed) how many times one has to go around γ.

By Proposition 2.2, δ is actually univalent, and since I is injective (when restricted to U_0), it coincides with the local δ above.

Remark. Let w be the dual 1-form to v. Then it is easy to see that the 1-form on $\mathfrak{G}$, $\Omega = \frac{1}{2}(p_1^*(w) - p_2^*(w))$, is closed in $\mathfrak{G}$, and it is just the differential of $\delta \colon \mathfrak{G} \to R$ by Proposition 2.3. In the case (excluded in this paper) where all orbits of v are closed, however, Ω can define a nontrivial element of $H^1(\mathfrak{G}, R)$.

3. Proof of Theorem I

In the following all geodesics will always be parametrized with respect to arc length. Let W be a complete riemannian manifold and $p\colon W \to R$ a riemannian submersion. Recall that then $|\operatorname{grad} p| \equiv 1$ and every orbit of $\operatorname{grad} p$ is a geodesic of W and conversely [9, p. 155]. We need the following results for whose proofs we give references when needed.

Lemma 3.1 (*see* [13, *p.* 262, *Theorem* 5.1]). *No fiber* $p^{-1}(\tau)$, $\tau \in R$, *has focal points.*

Let $F = p^{-1}(0)$ be totally geodesic in W. Let ψ^t denote the flow on W generated by the gradient of p, and γ_x the orbit of $\operatorname{grad} p$ through x.

Lemma 3.2. *For all* t, *any* $x \in f$ *and any unit vector* $u \in T_xF$, $d\psi^t(u) = J(t)$, *where* J *is the unique Jacobi field of* W *along* γ_x *such that* $J(0) = u$ *and* $\dot{J}(0) = 0$.

Proof. Let $\alpha(s)$ be a curve in F through x such that $\frac{d\alpha}{ds} = u$ for $s = 0$, and consider the variation $\rho(s, t) = \psi^t(\alpha(s))$. Then the Jacobi field $J = \frac{\partial\rho}{\partial s}(0, t)$ satisfies $J(0) = u$ and is normal to $\dot{\gamma}_x(t)$, and hence by [3, Proposition 3.6, p. 100] $\dot{J}(0)$ is also perpendicular to $\dot{\gamma}_x(0)$. However, since F is totally geodesic in W, by [3, Lemma 4.1, p. 181] (with $N = F$) $\dot{J}(0)$ is also parallel to $\dot{\gamma}_x(0)$, i.e., $\dot{J}(0) = 0$.

Combining this with a version of the Rauch Comparison Theorem [3, Theorem 4.7, p. 188] we obtain

Lemma 3.3. *Let* m_t *be the minimum of the sectional curvature of* W *along the orbit interval* $[\gamma_x(0), \gamma_x(t)]$ *of* γ_x *with respect to all 2-planes tangent to* γ_x, *and assume* $m_t \le 0$. *Then* $|d\psi^t(u)| \le \cosh t\sqrt{-m_t}$ *for every unit vector* $u \in T_xF$.

Here we have also used the following elementary fact (with $K = m_t$): On a manifold of constant nonpositive sectional curvature K, a Jacobi field J with the above properties satisfies $|J(t)| = \cosh t\sqrt{-K}$ with respect to any geodesic (see [6, p. 119]).

Denote by $\overline{\Phi}^t$ the flow of $\overline{V}$ on $\mathfrak{G}$. Since the diagonal D is a totally geodesic submanifold of $\mathfrak{G}$, by Proposition 2.3 we can apply Lemma 3.3 (with $\mathfrak{G} = W$, $\delta = p$, $D = F$, $\overline{\Phi}^t = \psi^t$) and obtain $|d\overline{\Phi}^t_{(x,x)}(z)| \le \cosh t\sqrt{-m_t}$ for any unit vector z tangent to the diagonal D.

Since $I: \mathfrak{G} \to M \times M$ is an isometric immersion we have $|d\overline{\Phi}| = |d\Phi|$, and since the following computations are local we assume $\mathfrak{G}$ is a submanifold of $M \times M$ with the induced riemannian metric.

We show $-m_t = \overline{E}_{xt}$ (see §1): let $Z \in T\mathfrak{G}$, let $\alpha(Z, V)$ be the second fundamental form of $\mathfrak{G}$ in $M \times M$, and let $\overline{K}_{ZV}$ and K_{ZV} be the sectional curvatures of $\mathfrak{G}$ and $M \times M$, with respect to the 2-plane spanned by Z and V. Since $\alpha(V, V) = 0$, the Gauss formula gives $-\overline{K}_{ZV} = |\alpha(Z, V)|^2 - K_{ZV}$, and if $\overline{\nabla}$ and ∇ denote the covariant derivatives in $\mathfrak{G}$ and $M \times M$, we have $|\nabla_Z V|^2 = |\overline{\nabla}_Z V|^2 + |\alpha(Z, V)|^2$ and $\langle V, \overline{\nabla}_Z V\rangle = 0$ (because $|V| \equiv 1$). Since the manifold $\overline{\Phi}^t(D)$ is a fiber of $\delta: \mathfrak{G} \to R$, i.e., perpendicular to $\overline{V}$, we get $\overline{\nabla}_Z V = \beta(Z, V)$ (see §1), i.e., $-\overline{K}_{ZV} = |\nabla_Z V|^2 - K_{ZV} - |\beta(Z, V)|^2$, and $-m_t = \overline{E}_{xt}$ follows.

To prove Theorem II simply drop the β term in Theorem I and compute in terms of M (instead of $M \times M$).

Finally, we need the following.

Lemma 3.4. *Let w be a unit vector field with geodesic orbits on the riemannian manifold N^n, and let f^t denote its flow. In addition, assume the $(n-1)$-plane field, $w^\perp$, perpendicular to w, is integrable, let a_j $(j = 1, \cdots, n-1)$ be a basis of $w^\perp$ at a point $x \in N$, and let A_t denote the $(n-1) \times (n-1)$ matrix $\{\langle df^t(a_i), df^t(a_j)\rangle\}$. Then $\mu_{xt} = \max\{-K_{zw}\}$ at $f^t(x)$ is equal to the maximum eigenvalue of the matrix $\frac{1}{2}\ddot{A}_t A_t^{-1} - \frac{1}{4}(\dot{A}_t A_t^{-1})^2$, where z ranges over all unit vectors perpendicular to w.*

Proof. Let y_j, $j = 1, \cdots, n-1$, be a chart of the leaf through x such that $\partial/\partial y_j = a_j$ at x. Consider the chart of N at x defined by $x_i = f^t(y_i)$ for $i < n$ and $x_n = t$, and set $X_i = \partial/\partial x_i$; thus $X_n = w$.

Let G_t denote the $n \times n$ matrix $\{g_{ij}\}$ at $f^t(x)$ of this chart, and observe that since $g_{nn} \equiv 1$ and $w^\perp$ is integrable, G_t is obtained from A_t by adding a 1 to the diagonal and 0's elsewhere, i.e., $g_{ni} = 0$ for $i < n$. This implies $\{\Gamma^s_{in}\} = \frac{1}{2}\dot{G}_t G_t^{-1}$ for all $i < n$; furthermore, since $\Delta_w w \equiv 0$, $\Gamma^s_{nn} = 0$.

Let R denote the curvature tensor of N; by a well-known elementary fact about matrices, $-\nu_{xt}$ above, i.e., the maximum of the quadratic form $\langle R(w, z)w, z\rangle$ (where $|z| = 1$, $z \in w^\perp$), is equal to the maximum eigenvalue of the matrix $\{R^s_{njn}\}$ defined by $R(w, X_j)w = R(X_n, X_j)X_n = \sum_s R^s_{njn} X_s$. Since

$$R^s_{njn} = \sum_l \Gamma^l_{nn}\Gamma^s_{jl} - \sum_l \Gamma^l_{jn}\Gamma^s_{nl} + \frac{\partial}{\partial x_j}\Gamma^s_{nn} - \frac{\partial}{\partial x_n}\Gamma^s_{jn}$$

(see [3, (2), p. 81]), the relations above imply

$$-R^s_{njn} = \frac{\partial}{\partial x_n}\Gamma^s_{jn} + \sum_l \Gamma^l_{jn}\Gamma^s_{nl} = \frac{1}{2}(\dot{G}_t G_t^{-1})^1 + \frac{1}{4}(\dot{G}_t G_t^{-1})^2$$
$$= \frac{1}{2}\ddot{G}_t G_t^{-1} - \frac{1}{4}(\dot{G}_t G_t^{-1})^2,$$

whose eigenvalues are the same as those of the $(n-1)\times(n-1)$ matrix $\frac{1}{2}\ddot{A}_t A_t^{-1} - \frac{1}{4}(\dot{A}_t A_t^{-1})^2$, and Lemma 3.4 is proven.

To obtain the remaining unproven result of §1 apply Lemma 3.4 with $N = \mathfrak{G}$, $w = \overline{V}$ and $f^t = \overline{\Phi}^t$.

4. A lower bound

Define $\bar{e}'_{xt}$ as in §1 except using the minimum (instead of the maximum) and let $\overline{E}'_{xt} = \min \bar{e}'_{x\tau}$ for $\tau \in [0, t]$; suppose $\overline{E}'_{xt} \geq 0$.

Theorem III. *For any $x \in M$ and any unit vector $u \in T_xM$, we have*

$$|d\varphi^t(u)|^2 + |d\varphi^{-t}(u)|^2 \geq 2\cosh^2 t\sqrt{2\overline{E}'_{x\sqrt{2}t}} \quad \text{for all } t \geq 0.$$

Proof. This is a straightforward consequence of our Lemma 3.2 (with $W = \mathfrak{G}$, $p = \delta$, $\psi^t = \overline{\Phi}^t$, $F = D$), inequality 4.10 of [5, Theorem 4.1′, p. 48] and our computations in §3 above.

Since $|\nabla_{\overline{Z}} V|^2 \geq |\beta(\overline{Z}, V)|^2$ Theorem IV follows immediately.

5. Examples

(i) Using the notation of the Introduction one easily shows (using formula (9), p. 47 of [3]) that at every $x \in M$

$$|\nabla_z v|^2 - K_{zv} = v\langle z, \nabla_z v\rangle + \langle z, \nabla_{[z,v]} v\rangle + \langle [z, v], \nabla_z v\rangle$$

for every unit vector $z \in T_xM$ perpendicular to v. Hence, using Killing's equation [3, p. 72] we obtain $e_x \equiv 0$ if v is also a Killing vector field.

Although, of course, this case is trivial, this formula relates, in general, the quantity e_x to the obstruction of v being Killing.

(ii) Let S denote the vector field which generates the geodesic flow on the unit sphere bundle, SM (provided with the Sasaki metric), of a riemannian manifold M. Then S is a unit vector field whose orbits are geodesics (see [10]).

First observe that the quantity $|\nabla_Z S|^2 - K_{ZS}$, where $Z \in TSM$, $\langle Z, S\rangle = 0$, is the same, whether we compute in SM or in TM, and so we use TM.

We use the notation [4, p. 76] and [7], which we assume the reader has at hand. (Notice that their curvature tensor R is the negative of ours.)

Let ξ be a unit vector belonging to T_xM; the following quantities of TTM are assumed to be computed at the point (x, ξ) of SM.

The unit normal vector field of SM in TM is given by $n = \sum \xi^i X_i^v$ and $S = \sum \xi^i X_i^h$. Using formulas (10) and (11) of [7, p. 125] and formulas (24) and (25) of [4, p. 79] one computes

$$\tilde{\nabla}_{U^h} S = -\tfrac{1}{2}[R(\xi, U)\xi]^v \quad \text{and} \quad \tilde{\nabla}_{U^v} S = U^h - \tfrac{1}{2}[R(\xi, U)\xi]^h$$

for any vector field U of M.

Let $Z \in TTM$, set $Z = Y_1^h + Y_2^v$, where Y_1 $Y_2 \in T_xM$, and notice if $\langle Z, S\rangle = 0$ and $Z \in TSM$ (i.e., $\langle Z, n\rangle = 0$) then $\langle Y_1, \xi\rangle = \langle Y_2, \xi\rangle = 0$ in T_xM.

Now let $\dim M = 2$ and, given $U \in TM$, let $\overline{U}$ denote a unit vector normal to U; consider the sectional curvature of M as a function $K: M \to R$.

Using formulas (18) and (21) of [7] and the fact that if $\dim M = 2$ then $\langle (\nabla_U R)(U, \overline{U})U, \overline{U}\rangle = -dK(U)$ for any unit vector field U of M, we obtain

$$|\tilde{\nabla}_Z S|^2 - K_{ZS} = K^2|Y_1|^2 + |Y_2|^2 + Y_1' Y_2' \, dK(\xi) - K$$

(where Y_1', Y_2' denote the numbers defined by $Y_1 = Y_1'\bar{\xi}$, $Y_2 = Y_2'\bar{\xi}$) from which the expression for e of S of SM^2 at the point (x, ξ_x) in the Introduction easily follows.

6. Remarks and questions

Our method is quite different and more general than those of [1], [2] and [8] which start off by assuming v is a geodesic flow. Furthermore, when applied to geodesic flows (at least on surfaces) our inequality in §5(ii) is sharper than the easily obtained inequality in [2, Appendix, p. 270].

Are our inequalities sharp enough to solve Osserman's problem [1, Problem 1.8, p. 6]?

Are Theorems II and IV really the sharpest 'static' corollaries of Theorems I and III? Even in the case of geodesic flows?

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